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Calculating Cross-Orthogonality with Expanded Test Data

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Introduction

VARIOUS means exist for assessing correlation between math model predictions and modal test data. One relatively simple way is the so-called cross-orthogonality (c/o) computation. The pre- and post-multiplication by test, ϕ_{test}, and predicted, ϕ_{anal}, modal vectors is generally performed on the mass matrix M as indicated in Eq. (1).

$$[c/o] = [\phi_{test}]^T[M][\phi_{anal}] \tag{1}$$

The c/o matrix is then examined for its orthogonality properties. While the concept is simple, cognizance of certain ramifications enhances the assessment of such data.

Compatibility

In general, math model dynamic degrees-of-freedom (DOF) will not be uniquely one-for-one with accelerometers placed upon the prototypes. Consequently, reduction of the math model will be required. If all accelerometers do not lie directly at math model DOF, this reduction will require more than one step. Following the standard force-free reduction, a second calculation must kinematically transform to juxtaposition. Once developed, of course, the double transformation may be combined into one.

This total transformation, applied to the math model mass matrix in a Guyan reduction,[1] satisfies compatibility requirements. The purpose of this Note is primarily to point out that more detailed information is retained if the mass matrix reduced in the above manner is *not* employed. Rather, the transformation (applied to the test modes) essentially provides for their expansion to math model size. Then the c/o computations will employ as independent quantities all DOF in the analytical modal vectors.

Normalization

Analytical modes used in this manner will already be correctly normalized. Test modes in their expanded form must be normalized in the same way, and supplemented by a null matrix.

Equation (2) shows the conventional transformations used to reduce the mass matrix.

$$\underset{N\times N}{\left[\, c/o \,\right]} = \underset{N\times N}{\left[\diagdown D \diagdown\right]}^{-1/2} \underset{N\times(A+O)}{\left[\, \phi_{test}^T \;\vdots\; 0 \,\right]} \underset{(A+O)\times A}{\left[\, T_2 \,\right]}^T \times \underset{A\times(A+O)}{\left[\, T_1 \,\right]}^T \underset{(A+O)\times(A+O)}{\left[\begin{array}{c:c} M_{AA} & M_{AO} \\ \hdashline M_{OA} & M_{OO} \end{array}\right]} \underset{(A+O)\times A}{\left[\, T_1 \,\right]} \times \underset{A\times(A+O)}{\left[\, T_2 \,\right]} \underset{(A+O)\times N}{\left[\begin{array}{c} \phi_{anal_A} \\ \hdashline \phi_{anal_O} \end{array}\right]} \tag{2}$$

where

A = the kept set, compatible with the test set
O = the omit set, or reduced set
N = the number of modes

$$T_1 = \left[\frac{I}{G_{OA}}\right] \text{ and } G_{OA} = -K_{OO}^{-1}K_{OA} \text{ (Ref. 1)}$$

T_2 = kinematic transformation
$= [I \;\vdots\; 0\,]$ for coincident test/analysis points
D = test mode normalization factors

Equation (3) shows the proposed method of expanding the test modes to math model size through the transformation $T_2^T T_1^T$.

$$\left[\, c/o \,\right] = \left[\diagdown D \diagdown\right]^{-1/2} \left[\, \phi_{test} \;\vdots\; 0 \,\right]^T \times \left[\, T_2 \,\right]^T \left[\, T_1 \,\right]^T \left[\, M \,\right] \left[\, \phi_{anal} \,\right] \tag{3}$$

Since dynamically the T_1 matrix will tend to introduce error through the static reduction term G_{OA} ($\phi_O \cong G_{OA}\phi_A$), the elimination of one T_1 term by Eq. (3) provides a more accurate and simplified solution.

Reference

[1]Guyan, R. J., "Reduction of Stiffness and Mass Matrices," *AIAA Journal*, Vol. 3, Feb. 1965, p. 380.

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Amplification of Discontinuities in a Radiating Gas

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IN our earlier paper,[1] using the optically thin approximation to the radiative transfer equation, we discussed the effects of radiative flux on the amplification or attenuation of finite amplitude waves propagating into a region assumed to be uniform and at rest. In a radiating gas, the validity of this assumption is questionable, because the tendency of a thermal precursor, which propagates with a velocity comparable to that of light, is, in general, to disturb the flow ahead of a modified gasdynamic wave, see, e.g., Lick,[2] Prasad,[3] and Helliwell.[4] Thus, in the study of wave propagation in a radiating gas, one should, in general, take into account the unsteady behavior of the flow ahead of the wave.

The purpose of the present Note is to reanalyze the problem considered in Ref. 1, keeping in view the unsteady behavior of the flow ahead of the wave surface. It is shown that along the bicharacteristic curves of the governing differential equations, the growth equation for a weak discontinuity reduces to a Bernoulli type equation, the solution of which has been completely analyzed in one of our papers.[5] Using the general

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results established in Ref. 5, we determine the time taken to reach shock formation, and present a few explicit criteria for the blow-up of discontinuities along the bicharacteristic direction.

Using matrix notation, the basic equations of Ref. 1 can be written as

$$\frac{\partial U}{\partial t}+A^i U_{,i}+B=0 \tag{1}$$

where U is a column vector with five components (ρ, u_i, p), B is a column vector with five components, and A^i are square matrices of order five, which can be read off by inspection of Eqs. (1-3) of Ref. 1. Let us now introduce the characteristic wave surface S, $\phi(X_i,t)=0$, across which the set of dependent variables U and the interior derivatives of U are continuous but the exterior derivatives of U may have a jump. Across such a surface, Eq. (1) yields

$$(I\phi_{,t}+A^i\phi_{,i})[U_{,\phi}]=0 \tag{2}$$

where I is a unit matrix of order five and the square brackets denote a jump in the quantity enclosed. Normalizing Eq. (2) by dividing by $|\nabla\phi|$ and setting $n_i=\phi_{,i}/|\nabla\phi|$, the unit normal to S, and $G=-\phi_{,t}/|\nabla\phi|$, the speed of propagation of S along n_i, we find that Eq. (2) will have a nontrivial solution if det $(A^i n_i - GI)=0$, which yields $G=u_n+c$ as one of the modes of wave propagation. Here u_n is the fluid velocity normal to S, and $c=(\gamma p/\rho)^{1/2}$ is the sound speed. Corresponding to the eigenvalue $G=u_n+c$ of $A^i n_i$, the right eigenvector of $A^i n_i$ yields the following relation

$$\lambda=\lambda^i n_i=\xi/(\rho_1 c_1)=c_1\zeta/\rho_1 \tag{3}$$

where subscript 1 denotes a value in the region ahead of S, and $\lambda_i=[\partial u_i/\partial n]$, $\xi=[\partial p/\partial n]$, and $\zeta=[\partial\rho/\partial n]$ are the discontinuities in the normal derivatives of the dependent variables. When the momentum and energy equations of Ref. 1 are differentiated with respect to x_k, and jumps are taken across $S(t)$, we find, using the compatibility conditions of Thomas,[6] that

$$\rho_1\frac{\delta\lambda}{\delta t}+\rho_1 u_1^\alpha\lambda_{,\alpha}+(\bar\xi-\rho_1 c_1\bar\lambda)-c_1\lambda\left(\frac{\partial\rho}{\partial n}\right)_1$$
$$+\left(\frac{\partial u_i}{\partial t}+u_j u_{i,j}\right)_1 n_i\zeta+2\rho_1\lambda\left(\frac{\partial u_i}{\partial n}\right)_1 n_i=0 \tag{4}$$

$$\frac{\delta\xi}{\delta t}+u_1^\alpha\xi_{,\alpha}-c_1(\bar\xi-\rho_1 c_1\bar\lambda)+(1+\gamma)\xi\lambda+\left\{\left(\frac{\partial u_i}{\partial n}\right)_1 n_i\right.$$
$$\left.+\gamma(u_{i,i})_1\right\}\xi+\lambda(1+\gamma)\left(\frac{\partial p}{\partial n}\right)_1-2\rho_1\lambda c_1^2\Omega$$
$$+4(p_1\gamma)^{-1}(\gamma-1)^2 Q_1\xi=0 \tag{5}$$

where $\delta/\delta t$ is the time derivative along an orthogonal trajectory of S, u_1^α $(\alpha=1,2)$ are the components of fluid velocity tangential to S, a comma followed by index α denotes partial derivative with respect to the surface coordinates y^α, Ω is the mean curvature of S, and $\bar\xi$ and $\bar\lambda$ are the quantities defined on S. The quantity Q is the rate of energy loss through radiation, as in Ref. 1.

Now let us recall that a bicharacteristic curve $x_i=x_i(t)$ of Eq. (1) satisfies the equation[7]

$$\frac{\mathrm{d}x_i}{\mathrm{d}t}=\frac{\partial F}{\partial\phi_{,i}}\Big/\frac{\partial F}{\partial\phi_{,t}} \tag{6}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t}=\frac{\partial}{\partial t}+v_i\frac{\partial}{\partial x_i}$$

is the time derivative moving with the wave front along the bicharacteristic curve with v_i as the characteristic velocity component and $F=|I\phi_{,t}+A^i\phi_{,i}|$ is the characteristic determinant of the system, which for the modified gasdynamic waves takes the form:

$$F=\phi_{,t}+u_i\phi_{,i}+c_1(\phi_{,j}\phi_{,j})^{1/2} \tag{7}$$

Equation (6), in view of Eq. (7), yields

$$v_i=\mathrm{d}x_i/\mathrm{d}t=u_i+c_1 n_i \tag{8}$$

Keeping in view the definition of the δ-time derivative and Eq. (8), it follows immediately that for any flow variable f defined on S,

$$\frac{\mathrm{d}f}{\mathrm{d}t}=\frac{\delta f}{\delta t}+u^\alpha f_{,\alpha} \tag{9}$$

Inserting the term $(\bar\xi-\rho_1 c_1\bar\lambda)$ from Eq. (5) into Eq. (4), and using Eqs. (3) and (9), we get

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t}+\{\ln(\rho_1 c_1)^{1/2}+P\}\lambda+\frac{(\gamma+1)}{2}\lambda^2=0 \tag{10}$$

where

$$P=\frac{3}{2}\left(\frac{\partial u_i}{\partial n}\right)_1 n_i+\frac{\gamma}{2}(u_{i,i})_1+\frac{(\gamma+1)}{2\rho_1 c_1}\left(\frac{\partial p}{\partial n}\right)_1-c_1\Omega$$
$$-\frac{c_1}{2\rho_1}\left(\frac{\partial\rho}{\partial n}\right)_1+(2c_1)^{-1}\left(\frac{\partial u_i}{\partial t}+u_j u_{i,j}\right)_1 n_i$$
$$+(p_1\gamma)^{-1}(2\gamma-1)^2 Q_1$$

Equation (10) is the required growth equation for the discontinuity λ, which we have been seeking. A complete analysis of the solution of such a Bernoulli type equation, modifying and generalizing several known results, can be found in Ref. 5. Equation (10) can be integrated to yield

$$\lambda=\frac{\lambda_0(\rho_1 c_1/\rho_{10}c_{10})^{-1/2}\exp(-J(t))}{1+(\rho_{10}c_{10})^{1/2}\lambda_0 I(t)} \tag{11}$$

where the subscript 0 indicates an initial value at $t=0$ and the functions I and J are given by

$$I(t)=\int_0^t\frac{(\gamma+1)}{2(\rho_1 c_1)^{1/2}}\exp(-J(\hat t))\,\mathrm{d}\hat t$$

$$J=\int_0^t P(\tau)\,\mathrm{d}\tau$$

Equation (11) gives the variation of discontinuity λ in the normal velocity gradient at the wave head. It is clear that the temporal behavior of λ will depend on the signs of λ_0, the initial discontinuity at the wave head, and the integral J which depends on the properties of gas under study.

If one considers the behavior of λ over the time interval $[0,T]$, it is clear from Eq. (11) that if I and J are continuous for $0\le t<T$ and have finite limits $I(T)$ and $J(T)$ as $t\to T$, then for $\lambda_0>0$ (which corresponds to an expansion wave) the right-hand side of Eq. (11) will not only remain continuous throughout $0 \le t < T$, but will also approach a finite limit as $t \to T$; this means that in this case the continuity of the gas motion is maintained in the time interval $[0,T]$ and the formation of a discontinuous front is not possible. If $\lambda_0 < 0$, which corresponds to a compressive wave, it is clear from Eq. (11) that there exists a positive critical value λ_c of the initial discontinuity λ_0 given by $\lambda_c = 1/(\rho_{10} c_{10} I(T))$, such that for $|\lambda_0| < \lambda_c$, the discontinuity $\lambda(t)$ will remain finite throughout $[0,T]$, i.e., the motion remains continuous and the formation of a discontinuous front is again not possible. For $|\lambda_0| > \lambda_c$, there exists a finite time $t_c < T$ given by the solution of $I(t_c) = -1/(\lambda_0 (\rho_{10} c_{10})^{1/2})$ such that $|\lambda| \to \infty$ as $t \to t_c$, i.e., the discontinuity grows without bound in a finite time which signifies the appearance of a shock wave at an instant t_c; for $|\lambda_0| = \lambda_c$, it is evident that $t_c = T$.

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References

[1] Singh, R. S. and Sharma, V. D., "Amplification of Finite Amplitude Waves in a Radiating Gas," *AIAA Journal*, Vol. 19, Feb. 1981, pp. 252-254.

[2] Lick, W., "Propagation of Small Disturbances in a Radiating Gas," *Journal of Fluid Mechanics*, Vol. 18, Pt. 2, 1964, pp. 274-284.

[3] Prasad, P., "Non-linear Waves in Radiation Gas Dynamics," *Journal of the Indian Institute of Science*, Vol. 49, 1967, pp. 148-178.

[4] Helliwell, J. B., "The Propagation of Small Disturbances in Radiative Magnetogasdynamics," *Archives of Rational Mechanics and Analysis*, Vol. 47, No. 5, 1972, pp. 380-388.

[5] Menon, V. V. and Sharma, V. D., "Characteristic Wave-Fronts in Magnetohydrodynamics," *Journal of Mathematical Analysis and Applications*, Vol. 81, May 1981, pp. 189-203.

[6] Thomas, T. Y., "Extended Compatibility Conditions for the Study of Surfaces of Discontinuity in Continuum Mechanics," *Journal of Mathematics and Mechanics*, Vol. 6, 1957, pp. 311-322.

[7] Courant, R. and Hilber, D., *Methods of Mathematical Physics*, Vol. II, Wiley Interscience, N.Y., 1962.

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